
UNIVERSITY OF WATERLOO
FACULTY OF ENGINEERING
Department of Electrical & Computer Engineering
ECE 204 Numerical methods
Approximating solutions to the heat and wave equations in two and three dimensions
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BY NC SA

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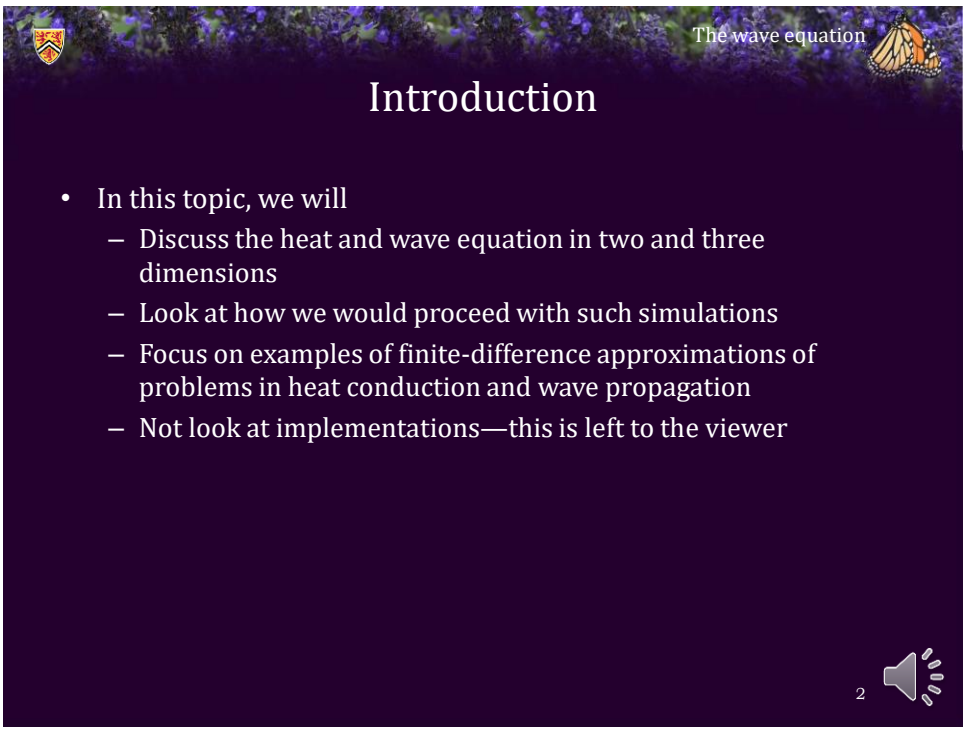
The wave equation
Introduction
• In this topic, we will
– Discuss the heat and wave equation in two and three dimensions
– Look at how we would proceed with such simulations
– Focus on examples of finite-difference approximations of problems in heat conduction and wave propagation
– Not look at implementations—this is left to the viewer
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The heat and wave equations

- We have approximated solutions to:
  - The heat and wave equations in one dimension
  - General boundary-value problems and Laplace's equation one, two and three dimensions
- We will now look at approximating solutions to the heat and wave equations in two and three dimensions

In two dimensions

- The heat and wave equations are

$$\frac{\partial}{\partial t}u(\mathbf{x},t)=\alpha\nabla^2 u(\mathbf{x},t) \qquad \frac{\partial^2}{\partial t^2}u(\mathbf{x},t)=c^2\nabla^2 u(\mathbf{x},t)$$

- For two dimensions, we will approximate $u(x_i, y_j, t_\ell)$ with $u_{i,j;\ell}$

$$u_{i,j;\ell+1}=u_{i,j;\ell}+\frac{\alpha\Delta t}{h^2}\left(u_{i-1,j;\ell}+u_{i+1,j;\ell}+u_{i,j-1;\ell}+u_{i,j+1;\ell}-4u_{i,j;\ell}\right)$$

$$u_{i,j;\ell+1}=2u_{i,j;\ell}-u_{i,j;\ell-1}+\left(\frac{c\Delta t}{h}\right)^2\left(u_{i-1,j;\ell}+u_{i+1,j;\ell}+u_{i,j-1;\ell}+u_{i,j+1;\ell}-4u_{i,j;\ell}\right)$$

In two dimensions

- First, we have a region over which heat is conducting or waves are propagating
  - Examples:
    - Heat conducting through a circuit board
    - Waves moving on water

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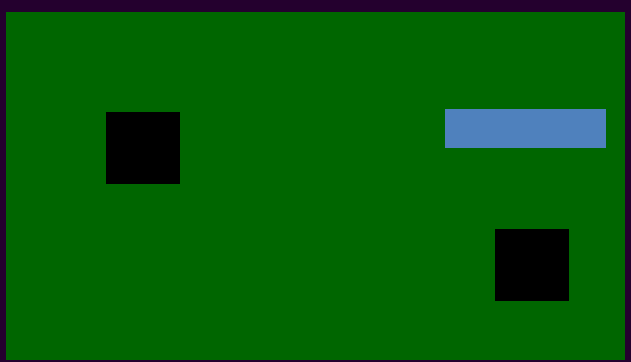
In two dimensions

- Here is a circuit board
  - The edges can be considered insulated



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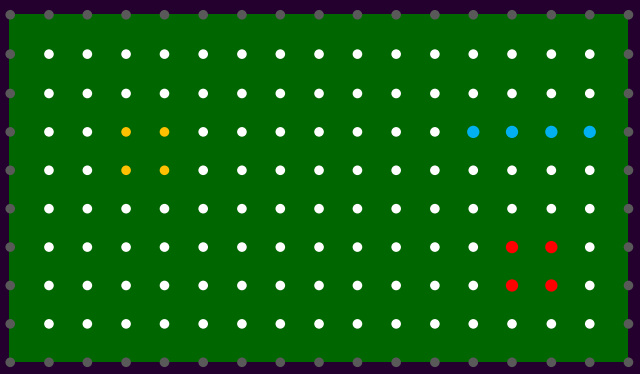
In two dimensions

- Here is a circuit board
  - The edges can be considered insulated



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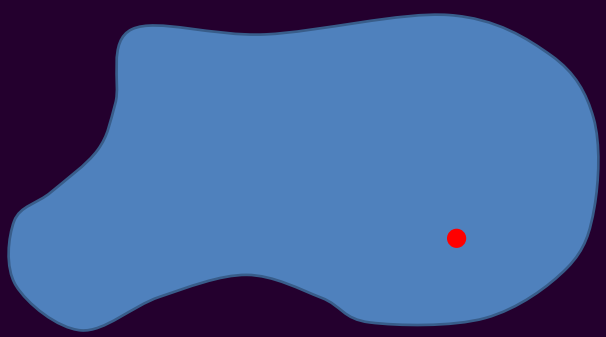
In two dimensions

- Here is a pool
  - The pool borders can be considered to be insulated



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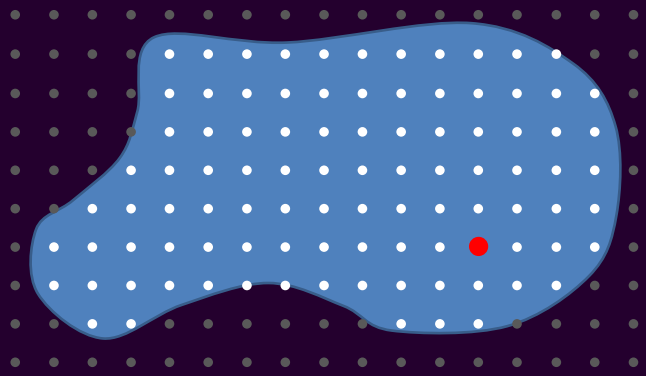
In two dimensions

- Here is a pool
  - The pool borders can be considered to be insulated



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In two dimensions

- If we know the temperature, voltage or wave amplitude at each point of the domain at time t_0, we will then use our finite-difference approximation of the corresponding equation to estimate the temperature, voltage or wave amplitude one time step into the future

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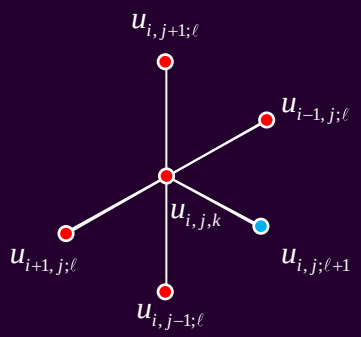
In two dimensions

- Visualizing this:

$$u_{i,j;\ell+1} = u_{i,j;\ell} + \frac{\alpha\Delta t}{h^2}\left(u_{i-1,j;\ell} + u_{i+1,j;\ell} + u_{i,j-1;\ell} + u_{i,j+1;\ell} - 4u_{i,j;\ell}\right)$$



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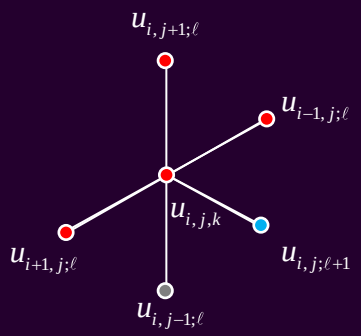
In two dimensions

- For insulated values, we assume $u_{i,j;\ell}$ equals the insulated value, and thus we eliminate that term

$$u_{i,j;\ell+1} = u_{i,j;\ell} + \frac{\alpha\Delta t}{h^2}\left(u_{i-1,j;\ell} + u_{i+1,j;\ell} + u_{i,j+1;\ell} - 3u_{i,j;\ell}\right)$$



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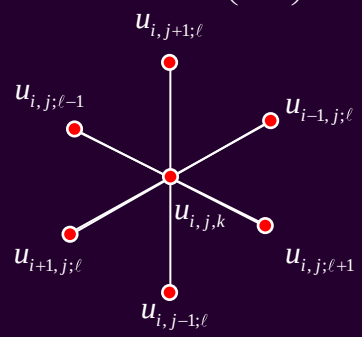
In two dimensions

- Visualizing this:

$$u_{i,j;\ell+1} = 2u_{i,j;\ell} - u_{i,j;\ell-1} + \left(\frac{c\Delta t}{h}\right)^2 \left(u_{i-1,j;\ell} + u_{i+1,j;\ell} + u_{i,j-1;\ell} + u_{i,j+1;\ell} - 4u_{i,j;\ell}\right)$$



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In three dimensions

- The heat and wave equations are

$$\frac{\partial}{\partial t}u(\mathbf{x},t) = \alpha\nabla^2 u(\mathbf{x},t) \qquad \frac{\partial^2}{\partial t^2}u(\mathbf{x},t) = c^2\nabla^2 u(\mathbf{x},t)$$

- For three dimensions,
  we will approximate $u(x_i, y_j, z_k, t_\ell)$ with $u_{i,j,k;\ell}$

$$u_{i,j,k;\ell+1} = u_{i,j,k;\ell} + \frac{\alpha\Delta t}{h^2}\left(u_{i-1,j,k;\ell} + u_{i+1,j,k;\ell} + u_{i,j-1,k;\ell} + u_{i,j+1,k;\ell} + u_{i,j,k-1;\ell} + u_{i,j,k+1;\ell} - 6u_{i,j,k;\ell}\right)$$

$$u_{i,j,k;\ell+1} = 2u_{i,j,k;\ell} - u_{i,j,k;\ell-1} + \left(\frac{c\Delta t}{h}\right)^2 \left(u_{i-1,j,k;\ell} + u_{i+1,j,k;\ell} + u_{i,j-1,k;\ell} + u_{i,j+1,k;\ell} + u_{i,j,k-1;\ell} + u_{i,j,k+1;\ell} - 6u_{i,j,k;\ell}\right)$$

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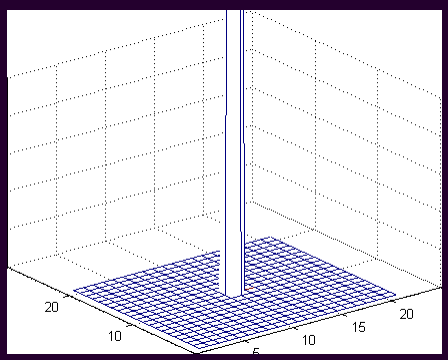
Example: heat equation in two dimensions

- Here we have a sheet of metal with the center heated to 1°C while the borders are maintained at 0°C
  - Over time, the heat distributes throughout the region and is absorbed in the boundaries



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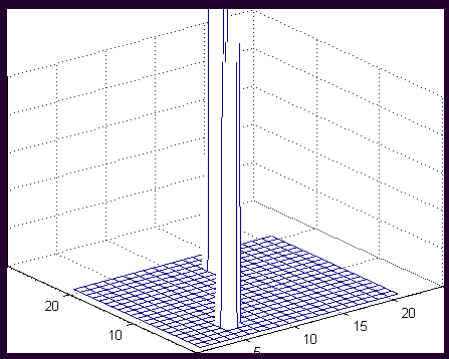
Example: heat equation in two dimensions

- Here we have a sheet of metal with two points heated to 5°C and 1°C while the borders are insulated
  - Over time, the heat distributes throughout the region the entire region approaches the average temperature



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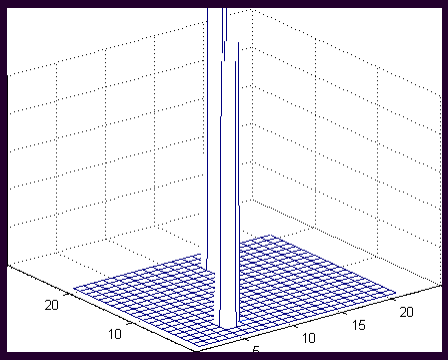
Example: heat equation in two dimensions

- Recall we required that

$$\frac{\alpha \Delta t}{h^2} < \frac{1}{2} \qquad \frac{c \Delta t}{h} < 1$$



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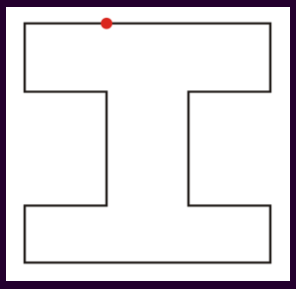
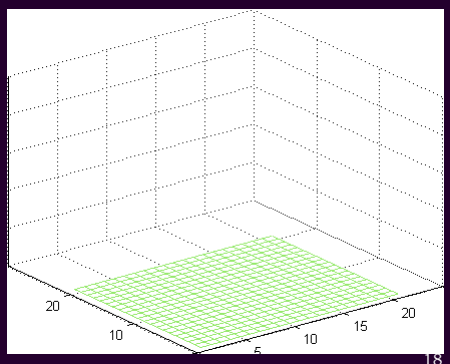
Example: heat equation in two dimensions

- Here is another example of such a problem:
  - Suppose we have an "H" shaped room with a heater
  - All other walls are insulated





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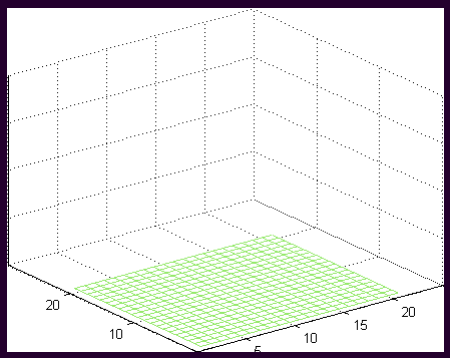
Example: heat equation in two dimensions

- Here we have an insulated room with one heat source at 20°C and one heat sink at 0°C
  - The walls are insulated and the temperature is initially 0°C



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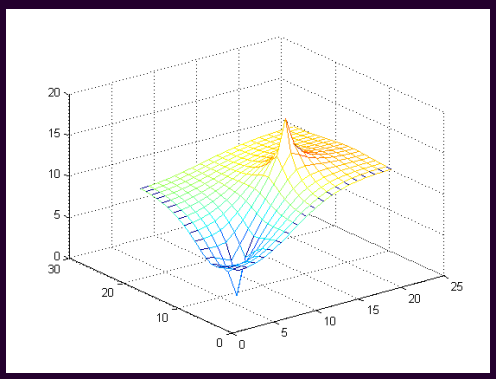
Example: heat equation in two dimensions

- The heat distribution converges to the solution to Laplace's equation shown here



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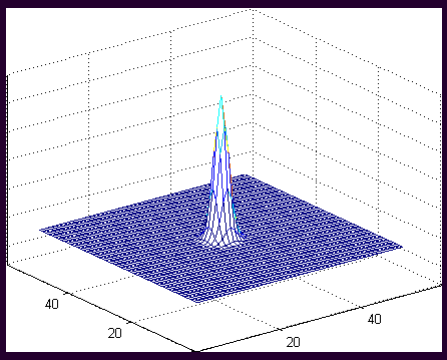
Example: wave equation in two dimensions

- Here we simulate a drop of water just coming into contact with the surface
  - The wave spreads out and strikes an insulated boundary



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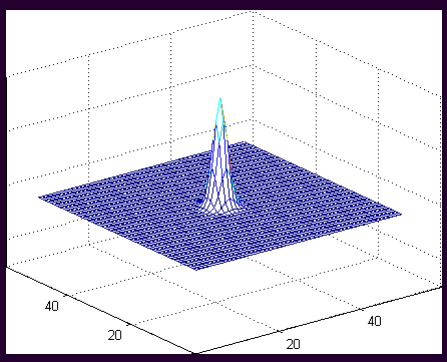
Example: wave equation in two dimensions

- Now the boundaries are circular but still insulated
  - The wave spreads out and strikes an insulated boundary simultaneously and bounce back



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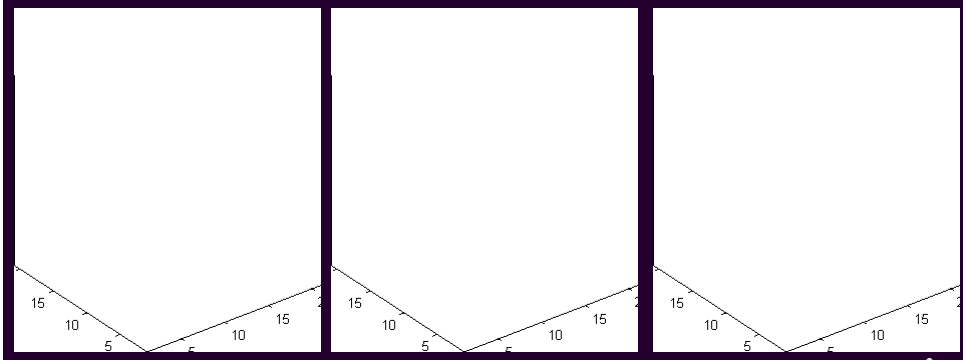
Example: heat equation in three dimensions

- Here we have a single point of heat, a line of heat and a wall of heat with all other walls insulated



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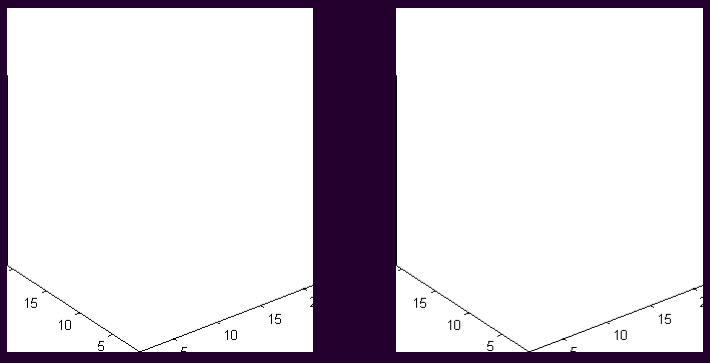
Example: wave equation in three dimensions

- Here is a spherical resonance chamber
  - A sinusoidal voltage is applied to the exterior
  - It is magnified at the center



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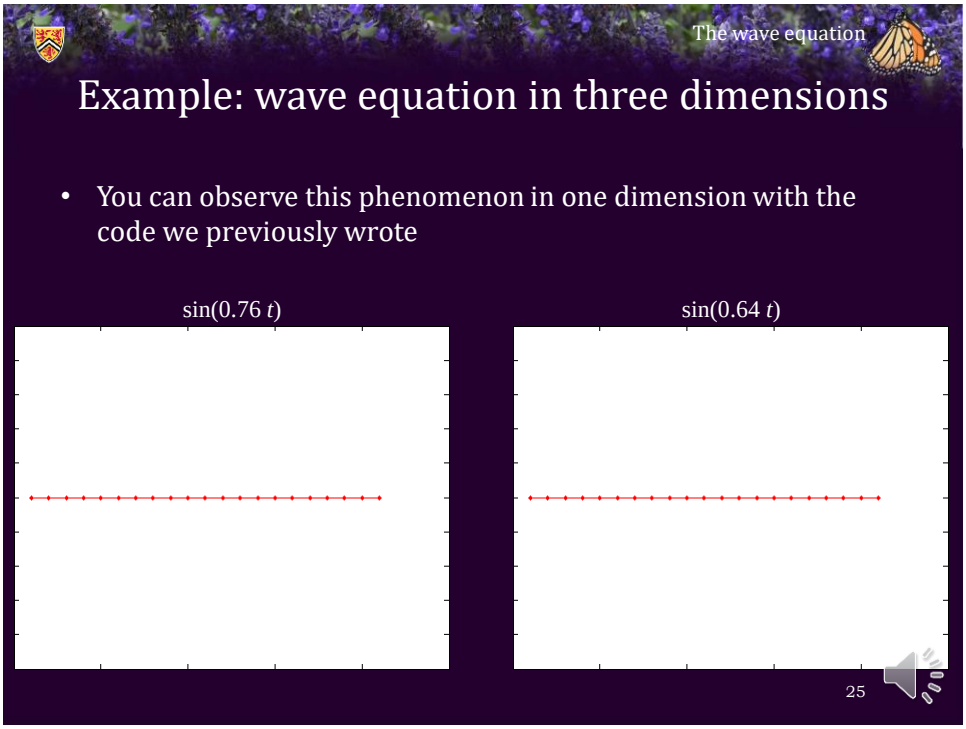
The wave equation
Example: wave equation in three dimensions
• You can observe this phenomenon in one dimension with the code we previously wrote
sin(0.76 t)
sin(0.64 t)
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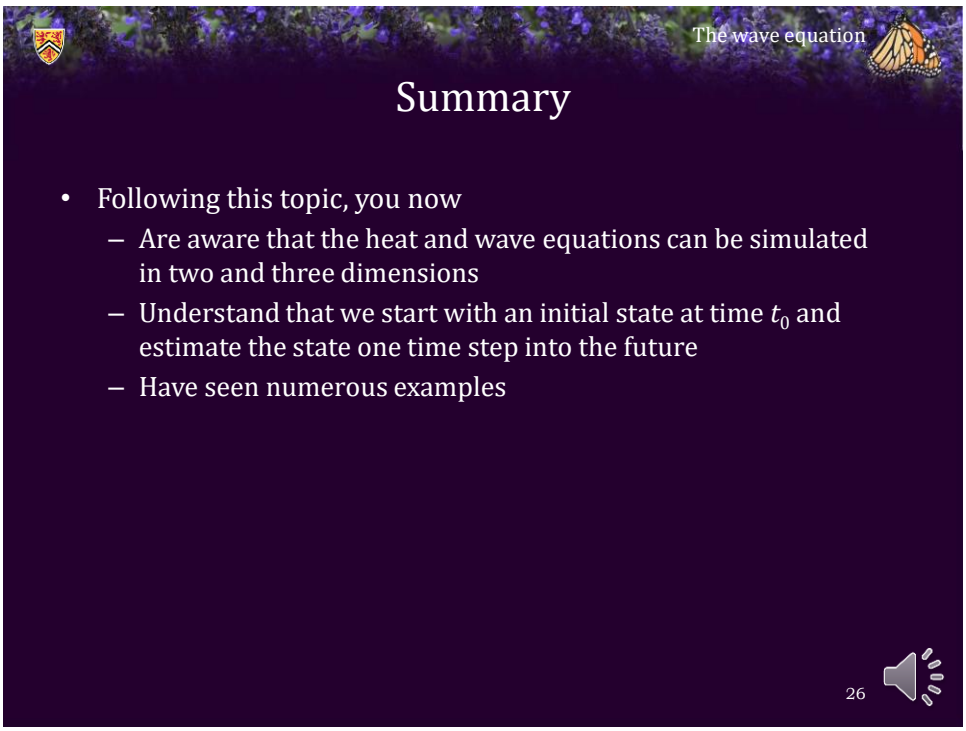
The wave equation
Summary
• Following this topic, you now
– Are aware that the heat and wave equations can be simulated in two and three dimensions
– Understand that we start with an initial state at time t_0 and estimate the state one time step into the future
– Have seen numerous examples
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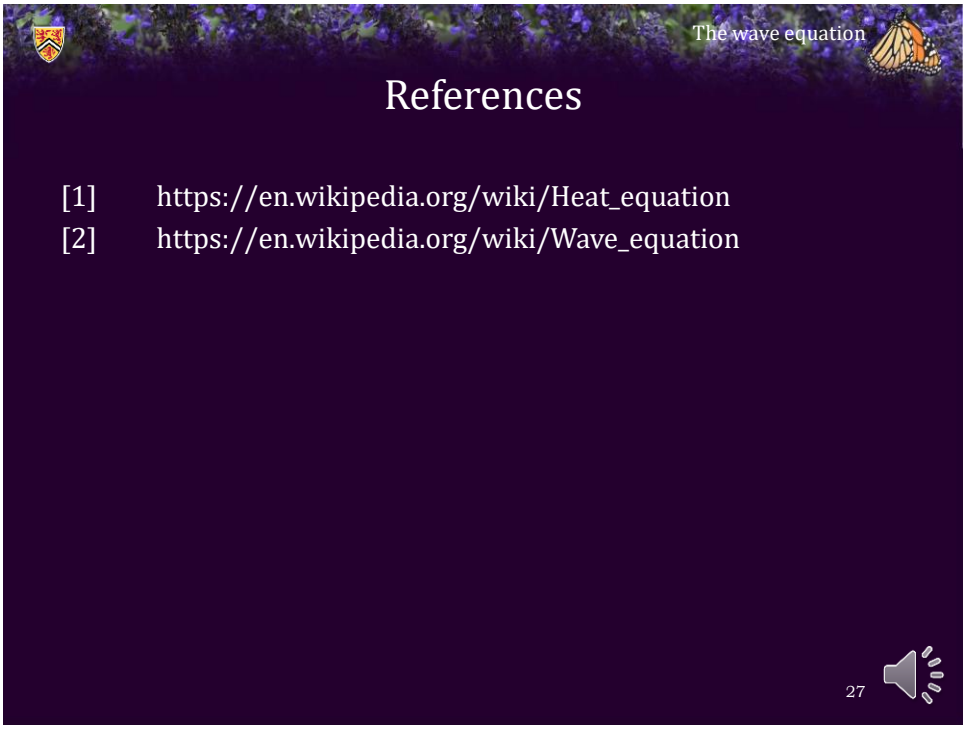
The wave equation
References
[1] https://en.wikipedia.org/wiki/Heat_equation
[2] https://en.wikipedia.org/wiki/Wave_equation
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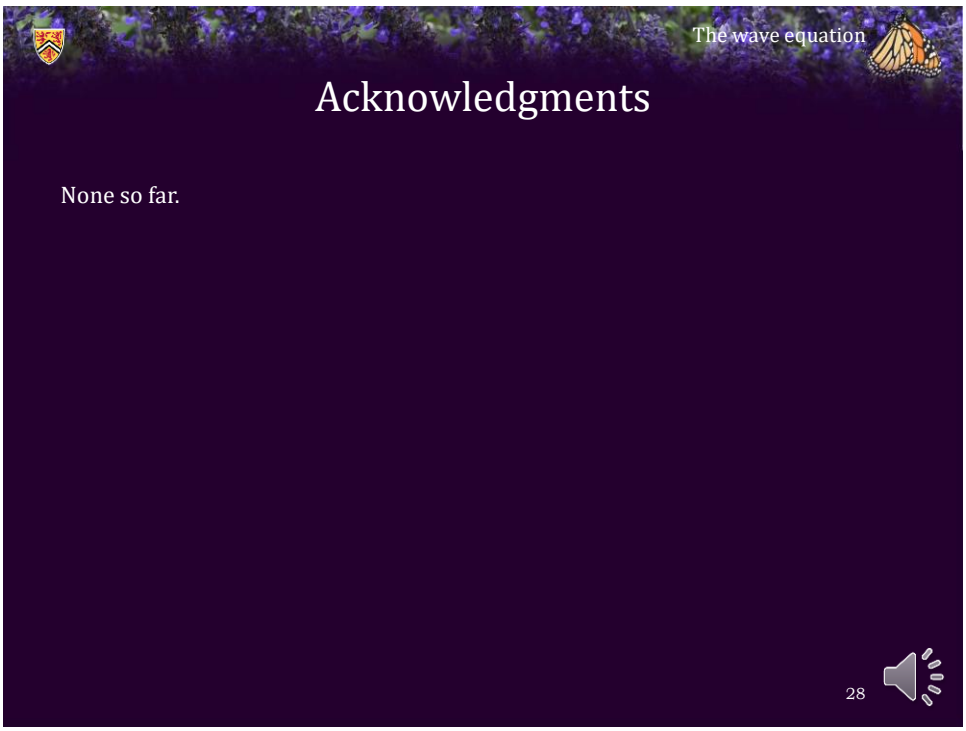
The wave equation
Acknowledgments
None so far.
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Colophon

These slides were prepared using the Cambria typeface. Mathematical equations use Times New Roman, and source code is presented using `Consolas`. Mathematical equations are prepared in MathType by Design Science, Inc.

Examples may be formulated and checked using Maple by Maplesoft, Inc.

The photographs of flowers and a monarch butter appearing on the title slide and accenting the top of each other slide were taken at the Royal Botanical Gardens in October of 2017 by Douglas Wilhelm Harder. Please see

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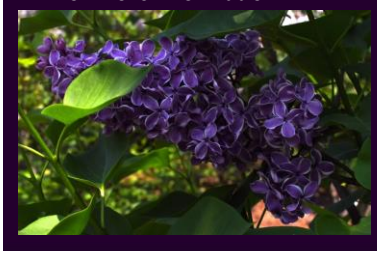
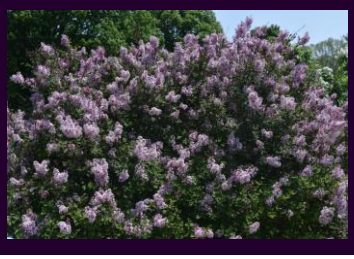
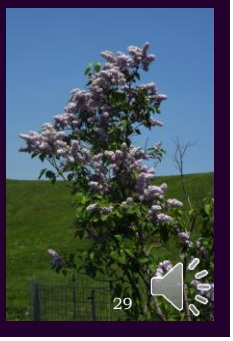
for more information.







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